A non-replicating proof of the Modigliani-Miller Theorem

We begin by proving a foundational result:

Lemma 1 *If an unlevered firm generates a constant cash flow x annually in perpetuity, its value equals $\frac{x}{r}$, where r is the risk-free rate.*

Proof. Lemma 1 might appear obvious at first glance. Intuitively, the value of a company—or any asset—should equal the present value of the cash flows it generates (Property 1). Hence, this is a special case of that general principle. Although the perpetuity formula is often derived as a consequence of the Modigliani-Miller (MM) theorem, we will instead verify it independently and then use it to prove MM.

Consider an unlevered firm A with no debt. Let $V_A = S_A$ denote the firm's value (equal to its equity). Suppose it generates perpetual annual cash flow x. Let r be the risk-free interest rate. We want to show:

$$V_A = S_A = \frac{x}{r}$$

Let N_A be the number of outstanding shares. Then the dividend per share is:

$$d = \frac{x}{N_A}$$

And the price per share is:

$$p_A = \frac{S_A}{N_A}$$

We consider two cases:

Case 1: $d < p_A r$. Then the dividend per share is lower than what one could earn by depositing p_A in the bank. Investors would sell shares and move their capital to the bank, pushing p_A down until $d = p_A r$.

Case 2: $d > p_A r$. Then even the firm itself would borrow p_A at rate r to repurchase its own shares—earning d and paying only $p_A r$. Since the firm is unlevered, this contradicts our assumption.

Thus, in equilibrium:

$$d = p_A r \Rightarrow \frac{x}{N_A} = \frac{S_A}{N_A} r \Rightarrow x = S_A r \Rightarrow S_A = \frac{x}{r}$$

□

Main Proof: Modigliani-Miller Theorem

Let firm 1 be unlevered with value $V_1 = S_1$ and firm 2 be levered with value $V_2 = S_2 + D_2$. Assume both firms generate the same perpetual cash flow x.

From Lemma 1, we have:

$$S_1 = \frac{x}{r}$$

Assume, for contradiction, that:

$$V_2 = S_2 + D_2 > V_1 = S_1$$

Let an investor currently hold β shares of firm 2. Then the investor's annual income is:

$$c = \beta \cdot \frac{x - rD_2}{N_2}$$

(The firm pays rD_2 in interest to bondholders; equityholders receive the residual.)

If the investor sells those shares (valued at $\beta p_2 = \beta S_2/N_2$) and buys shares of firm 1 (with price $p_1 = S_1/N_1$), he can buy:

$$i = \frac{\beta S_2 N_1}{N_2 S_1}$$

shares of firm 1, earning:

$$d = i \cdot \frac{x}{N_1} = \frac{\beta S_2 x}{N_2 S_1}$$

To find arbitrage, compare this with the original return:

$$\frac{\beta S_2 x}{N_2 S_1} > \frac{\beta(x - rD_2)}{N_2} \Rightarrow \frac{S_2 x}{S_1} > x - rD_2 \Rightarrow \frac{S_2 x}{S_1} - x > -rD_2 \Rightarrow x\left(\frac{S_2 - S_1}{S_1}\right) > -rD_2 \Rightarrow S_2 > S_1 - \frac{rD_2 S_1}{x}$$

But since $x = rS_1$ by Lemma 1:

$$S_2 > S_1 - D_2 \Rightarrow V_2 = S_2 + D_2 > S_1$$

which contradicts the equilibrium. Thus, no arbitrage can persist unless:

$$V_1 = V_2$$

Conclusion: We've shown that if two firms have identical perpetual cash flows, their values must be equal—regardless of capital structure. Notably, this proof avoids the "homemade leverage" argument and does not require the investor to borrow externally to replicate the firm's capital structure.

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